A Linear-Operator Formalism for the Analysis of Inhomogeneous Biisotropic Planar Waveguides

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Abstract—**Using the theory of linear operators, guided electromagnetic wave propagation in inhomogeneous (non-reciprocal) biisotropic planar structures is analyzed in terms of a 2 × 2 matrix differential operator. Based on the concept of adjoint waveguide, a new bi-orthogonality relation for the guided hybrid modes is derived. For the special case of *reciprocal biisotropic* media or *chiral* media, the linear-operator formalism leads to a self-adjoint problem. As an example of application, a general analysis of the radiation modes of a grounded chiroslabguide is also presented.**

I. Introduction

A BIANISOTROPIC medium with isotropic dependence among the field vectors is called a biisotropic medium [1]. This type of medium was conceived by Tellegen in 1948 in order to introduce the gyrator [2]. Since then biisotropic media have also been referred to as gyrational or Tellegen.

Chirality is a phenomenon known in optics since the beginning of the 19th century: in 1811 Arago discovered that crystals of quartz are optically active [3]. Later, in 1848, Pasteur postulated that optical activity was caused by the lack of symmetry (or chirality) of some molecules [4]. Indeed, chirality is a geometric concept related to objects which cannot be brought into congruence with their enantiomorphs [5].

Due to recent advances in polymer science and in the manufacture of artificial dielectrics, it is conceivable that chiral materials for applications at microwave or millimeter wavelengths can be made [5]. These advances have prompted renewed interest on electromagnetic wave propagation and radiation in chiral media.

A chiral medium, as described by some researchers (e.g., [5], [6]), is a reciprocal medium characterized by a three-parameter model. However, as mentioned by Monzon in [7] the four-parameter (or biisotropic) medium is also chiral in nature and obviously more general. Nevertheless, in order to avoid any risk of confusion, only the three-parameter medium will be called chiral herein.

In this paper, based on Maxwell's curl equations for source-free regions together with the set of constitutive relations for biisotropic media in the ***EH*** representation [1], a linear-operator formalism for the analysis of inhomogeneous biisotropic planar waveguides is presented. In fact, for these (*open or closed*) waveguides, using the theory of linear operators and through a suitable definition of a two-vector transverse mode function, the problem of guided electromagnetic wave propagation is reduced to an eigenvalue equation related to a 2 × 2 matrix differential operator. Introducing the concept of adjoint waveguide [8], a new bi-orthogonality relation [9] for the guided hybrid modes is derived. One should note that this linear-operator formalism is applicable to multilayered waveguides with *inhomogeneous and non-reciprocal* biisotropic layers. However, no specific example of application for this general case is worked out. Further investigations are needed in this area. Although the mathematical framework of this paper also uses the *theory of linear operators* as the one developed by the authors for the analysis of anisotropic layered waveguides [10], the present formalism is more general since it leads to a *non-self-adjoint* problem due to the non-reciprocal nature of material characteristics.

For multilayered waveguides with *homogeneous layers*, the general formalism is reduced to a 2 × 2 coupling matrix eigenvalue problem which will be presented elsewhere [11] and which is also similar to the analytical approach developed for anisotropic layered waveguides presented in [12].

For the special case of *chiral* (reciprocal) media, the linear-operator formalism leads to a *self-adjoint* problem. Therefore, with the theory developed in Section II and for *closed waveguides* (regular problems), the completeness of the set of eigenfunctions can be rigorously guaranteed due to self-adjointness [9]. Following a different approach, Engheta and Pelet also have presented several orthogonality relations for (reciprocal) chirowaveguides [13].

Finally, as an example of application, a general analysis of the radiation modes of a grounded chiroslabguide is also presented. One should note that, as far as the authors are aware, this problem has never been addressed in the literature—in spite of its potential interest for millimeter-wave structures. The complementary analysis of the surface modes in a grounded chiroslabguide is skipped over since it was already presented by the authors in [11], although this was on the basis of a different method which is unable to establish any orthogonality relation. More-

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over, following different approaches, the surface modes of a symmetric chiroslabguide were also analyzed in [14] and [15]. Analyses for the parallel-plate chirowaveguide [16], [17], the circular metal chirowaveguide [18], and the circular dielectric chirowaveguide [18], have also been presented.

II. Linear-Operator Formalism

The aim of this section is to reduce the problem of guided electromagnetic wave propagation in (*open* or *closed*) inhomogeneous (non-reciprocal) biisotropic planar waveguides to a linear-operator formalism. According to this formalism, linear independence of transverse mode functions is guaranteed. Therefore, assuming completeness, an arbitrary electromagnetic field in a biisotropic planar waveguide can be expanded in terms of these transverse mode functions.

For biisotropic media the constitutive relations may be written as [1], [7]

$$\boldsymbol{D} = \epsilon_0(\epsilon \boldsymbol{E} + Z_0 \xi \boldsymbol{H}) \tag{1a}$$

$$\boldsymbol{B} = \mu_0(Y_0 \zeta \boldsymbol{E} + \mu \boldsymbol{H}) \tag{1b}$$

where ϵ, μ, ξ and ζ are *dimensionless* parameters and $Z_0 = Y_0^{-1} = k_0/(\omega\epsilon_0) = (\omega\mu_0)/k_0$.

For the particular case of *reciprocal media* the pseudoscalars (ξ, ζ), which are responsible for *spatial dispersion* in biisotropic media, are such that $\xi + \zeta = 0$ [7].

Introducing normalized distances marked with primes (e.g., $x' = k_0 x$, $y' = k_0 y$, $z' = k_0 z$) and a normalized "magnetic field" $\mathcal{H}$ such that

$$\mathcal{H} = Z_0 \boldsymbol{H} \tag{2}$$

then, from Maxwell's curl equations for source-free regions together with (1a) and (1b), one may write

$$-j\nabla' \times \mathcal{H} = \epsilon \boldsymbol{E} + \xi \mathcal{H} \tag{3a}$$

$$j\nabla' \times \boldsymbol{E} = \zeta \boldsymbol{E} + \mu \mathcal{H} \tag{3b}$$

where time-harmonic field variation of the form $\exp(j\omega t)$ is adopted and $\nabla = k_0 \nabla'$.

Only planar structures which are uniform and infinite in the y direction (hence $\partial/\partial y = 0$) and inhomogeneously filled with biisotropic media in the x direction will be analyzed. More precisely: $\epsilon(x')$, $\mu(x')$, $\xi(x')$ and $\zeta(x')$ are piecewise-continuous functions of x' (i.e., the general case of inhomogeneous layers is included). Moreover, plane wave propagation of the form $\exp(-j\beta z')$ will be considered; β is an effective refractive index given by

$$\beta = \frac{k}{k_0} \tag{4}$$

where k is the longitudinal wavenumber. Hence, in (3a) and (3b), one has

$$\nabla' = \partial_{x'}\hat{x} - j\beta\hat{z} \tag{5}$$

where $\partial_{x'}$ stands for $\partial/\partial x'$.

A. Eigenvalue Equations for Inhomogeneous Waveguides

In order to recast the electromagnetic field equations in terms of a single eigenvalue equation, the following definition of a two-vector transverse mode function (or eigenfunction) is introduced:

$$\boldsymbol{\Phi} = [E_y, \mathcal{H}_y]^T \tag{6}$$

where superscript ^{T} stands for transpose. Therefore, from (1)–(6), one obtains the eigenvalue equation

$$\overline{\overline{\mathcal{L}}} \cdot \boldsymbol{\Phi} = \beta^2 \overline{\overline{\mathcal{W}}} \cdot \boldsymbol{\Phi} \tag{7}$$

which is formally similar to [10, eq. (19)]. The linear differential operator $\overline{\overline{\mathcal{L}}}$ is given by

$$\overline{\overline{\mathcal{L}}} = \begin{bmatrix} \partial_{x'} \dfrac{\epsilon}{\Delta} \partial_{x'} + \epsilon & \partial_{x'} \dfrac{\zeta}{\Delta} \partial_{x'} + \xi \\ -\partial_{x'} \dfrac{\xi}{\Delta} \partial_{x'} - \zeta & -\partial_{x'} \dfrac{\mu}{\Delta} \partial_{x'} - \mu \end{bmatrix} \tag{8}$$

and the "weight" operator $\overline{\overline{\mathcal{W}}}$ by

$$\overline{\overline{\mathcal{W}}} = \frac{1}{\Delta} \begin{bmatrix} \epsilon & \zeta \\ -\xi & -\mu \end{bmatrix} \tag{9}$$

where

$$\Delta = \epsilon\mu - \xi\zeta. \tag{10}$$

One should note, once again, that (7) is applicable to non-reciprocal biisotropic multilayered waveguides with inhomogeneous layers (i.e., $\epsilon(x')$, $\mu(x')$, $\xi(x')$ and $\zeta(x')$ may *continuously* vary inside each layer).

According to Appendix I, once the field components E_y and $\mathcal{H}_y$ have been determined through (7), the remaining field components can also be determined.

The minus sign in the second row of $\overline{\overline{\mathcal{L}}}$ and $\overline{\overline{\mathcal{W}}}$ will be useful in the reciprocal case (i.e., for $\xi = -\zeta$) as it allows the symmetry of these 2×2 matrices. One should also note that the case in which $\Delta = 0$ (i.e., $\epsilon\mu = \xi\zeta$) will be disregarded since an infinitesimally small imaginary part of ϵ can always ensure $\Delta \neq 0$.

According to (6)–(8) only hybrid modes can propagate in the planar structure, unless $\xi = \zeta = 0$ everywhere in which case TE and/or TM modes can propagate.

In everything that follows within this section, three classes of waveguides will be considered: (i) *closed* waveguides with electric and/or magnetic walls placed at $x' = 0$ and $x' = d'$; (ii) *open* waveguides extending from $x' = -\infty$ to $x' = \infty$; (iii) *open grounded* waveguides extending from an electric or magnetic wall placed at $x' = 0$ to $x' = \infty$. Hence, a finite, infinite, or semi-infinite interval I on x' will be introduced as follows: (i) $I = [0, d']$ for closed waveguides; (ii) $I =]-\infty, \infty[$ for open waveguides; (iii) $I = [0, \infty[$ for open grounded waveguides. In order to define the domain of $\overline{\overline{\mathcal{L}}}$, D, only surface modes will be considered for the two classes (ii) and (iii) of open waveguides. Consequently, E_y and $\mathcal{H}_y$ always have finite energy and hence they belong to the vector space of square integrable functions over I. However, only a complete spectral representation for closed waveguides (i.e., for regular problems corresponding to a finite interval I) is possible within D.

At this point it is convenient to introduce the concept of *adjoint* waveguide [8] as the one which has the same geometry and dimensions of the original waveguide but filled with biisotropic media characterized by ϵ^a, μ^a, ξ^a and ζ^a such that, for any $x' \in I$, one has

$$\epsilon^a = \epsilon \qquad \mu^a = \mu \tag{11a}$$

$$\xi^a = -\zeta \quad \zeta^a = -\xi. \tag{11b}$$

Therefore, one obtains for the adjoint waveguide

$$\overline{\mathfrak{L}}^a \cdot \mathbf{\Phi}^a = \beta_a^2 \overline{\mathfrak{W}}^a \cdot \mathbf{\Phi}^a \tag{12}$$

where, according to (8), (9), and (11), one can easily see that

$$\overline{\mathfrak{L}}^a = \overline{\mathfrak{L}}^T \quad \overline{\mathfrak{W}}^a = \overline{\mathfrak{W}}^T. \tag{13}$$

B. Bi-Orthogonality Relation

Denoting by D^a the domain of $\overline{\mathfrak{L}}^a$, the following real-type inner product can be introduced:

$$\langle \boldsymbol{u}, \boldsymbol{u}^a \rangle = \int_I (u_1 u_1^a + u_2 u_2^a)\, dx' \tag{14}$$

where $\boldsymbol{u} = [u_1, u_2]^T \in D$ and $\boldsymbol{u}^a = [u_1^a, u_2^a]^T \in D^a$. One should note that a full characterization of both D and D^a requires that all the expressions to which operator $\partial_{x'}$ is applied have to be continuous over I [10]. One can readily show that this requirement is equivalent to the continuity of the field components that are perpendicular to the x axis (i.e., to the inhomogeneity direction).

Using definition (14), one can prove that $\overline{\mathfrak{L}}^a$ and $\overline{\mathfrak{W}}^a$ are the adjoint operators [9] of $\overline{\mathfrak{L}}$ and $\overline{\mathfrak{W}}$, respectively, i.e.,

$$\langle \overline{\mathfrak{L}} \cdot \boldsymbol{u}, \boldsymbol{u}^a \rangle = \langle \boldsymbol{u}, \overline{\mathfrak{L}}^a \cdot \boldsymbol{u}^a \rangle \tag{15a}$$

$$\langle \overline{\mathfrak{W}} \cdot \boldsymbol{u}, \boldsymbol{u}^a \rangle = \langle \boldsymbol{u}, \overline{\mathfrak{W}}^a \cdot \boldsymbol{u}^a \rangle. \tag{15b}$$

The proof of (15a) can be found in Appendix II; the proof of (15b) is trivial and therefore will be omitted. According to these properties and to the fact that every eigenvalue β^2 of $\overline{\mathfrak{L}}$ is an eigenvalue of $\overline{\mathfrak{L}}^a$ [9], one can readily prove that

$$(\beta_m^2 - \beta_n^2)\langle \overline{\mathfrak{W}} \cdot \mathbf{\Phi}_m, \mathbf{\Phi}_n^a \rangle = 0 \tag{16}$$

if $\mathbf{\Phi}_m \in D$ and $\mathbf{\Phi}_n^a \in D^a$. Hence, after a suitable normalization, the following biorthogonality relation holds:

$$\langle \overline{\mathfrak{W}} \cdot \mathbf{\Phi}_m, \mathbf{\Phi}_n^a \rangle = \delta_{mn} \tag{17}$$

where δ_{mn} is the Kronecker delta. In fact, if $m \neq n$, $\beta_m^2 \neq \beta_n^2$ whenever $\xi(x')$ or $\zeta(x')$ are not identically null. According to the expressions given in Appendix I, (17) can also be written as

$$\int_I (E_{xm} \mathcal{H}_{yn}^a + \mathcal{H}_{xm} E_{yn}^a)\, dx' = -\beta_m \delta_{mn} \tag{18}$$

where $(E_{xm}, \mathcal{H}_{xm})$ are field components of the original waveguide whereas $(E_{yn}^a, \mathcal{H}_{yn}^a)$ are field components of the adjoint waveguide.

Only in the *reciprocal case* ($\xi = -\zeta$) the original and adjoint waveguides are identical: $D = D^a$, $\overline{\mathfrak{L}} = \overline{\mathfrak{L}}^a$, and hence the operator $\overline{\mathfrak{L}}$ is self-adjoint. In this case the bi-orthogonality relation (18) is reduced to an orthogonality relation in which $E_{yn}^a = E_{yn}$ and $\mathcal{H}_{yn}^a = \mathcal{H}_{yn}$. One should note that this last orthogonality relation could be alternatively derived from [13, eq. (23)] if the reflection symmetry [19, p. 231] for chirowaveguides was used, although without the formal simplicity of the spectral theory of linear operators. Nevertheless, the bi-orthogonality relation (18) is valid for the non-reciprocal case and does not appear to have been presented elsewhere.

For *closed* (non-reciprocal) biisotropic waveguides and assuming completeness, an arbitrary electromagnetic field characterized by $\mathbf{\Phi}(x')$ can be expanded in terms of the complete set of eigenfunctions $\{\mathbf{\Phi}_n(x')\}$ as follows:

$$\mathbf{\Phi}(x') = \sum_{n=1}^{\infty} \alpha_n \mathbf{\Phi}_n(x') \tag{19}$$

where, according to (17),

$$\alpha_n = \langle \overline{\mathfrak{W}} \cdot \mathbf{\Phi}, \mathbf{\Phi}_n^a \rangle. \tag{20}$$

One should also note that, for closed (reciprocal) chirowaveguides, the expansion (19) is rigorously guaranteed due to the self-adjointness of the operator [9].

For *open* biisotropic waveguides the complete set of modes is made up of a finite number of surface modes (the proper eigenfunctions of $\overline{\mathfrak{L}}$) together with a continuum of pseudosurface modes (the improper eigenfunctions of $\overline{\mathfrak{L}}$) [20], [9]. A grounded chiroslabguide, which will be analyzed in the next section, is an example of an open waveguide.

C. Homogeneous Layers

For the special case of *homogeneous* layers, the linear-operator formalism herein derived is reduced to a 2×2 coupling matrix eigenvalue problem. In fact, for this case, one obtains from (7)–(9)

$$\partial_{x'}^2 \mathbf{\Phi} = -\overline{\boldsymbol{C}} \cdot \mathbf{\Phi} \tag{21}$$

where [11]

$$\overline{\boldsymbol{C}} = \begin{bmatrix} \epsilon\mu - \zeta^2 - \beta^2 & \mu(\xi - \zeta) \\ -\epsilon(\xi - \zeta) & \epsilon\mu - \xi^2 - \beta^2 \end{bmatrix}. \tag{22}$$

Hence, in a similar way as shown in [11], [12], one may write for each homogeneous biisotropic layer:

$$\mathbf{\Phi}(x') = \overline{\boldsymbol{M}} \cdot \mathbf{\Psi}(x') \tag{23}$$

with

$$\overline{M} = \begin{bmatrix} 1 & 1 \\ \tau_1 & \tau_2 \end{bmatrix} \tag{24}$$

and where $\Psi = [\psi_1, \psi_2]^T$ is such that

$$\partial_{x'}^2 \Psi = -\text{diag}\,(\sigma_1^2, \sigma_2^2) \cdot \Psi \tag{25}$$

with ($s = 1, 2$)

$$\tau_s = -\left(\frac{\xi + \zeta}{2\mu}\right) \pm j\sqrt{\frac{\epsilon}{\mu} - \left(\frac{\xi + \zeta}{2\mu}\right)^2} \tag{26a}$$

$$\sigma_s^2 = \beta_\pm^2 - \beta^2 \tag{26b}$$

$$\beta_\pm = \sqrt{\epsilon\mu - \left(\frac{\xi + \zeta}{2}\right)^2} \pm j\left(\frac{\xi - \zeta}{2}\right). \tag{26c}$$

The *plus* (resp. *minus*) sign corresponds to $s = 1$ (resp. $s = 2$).

For an unbounded biisotropic medium there are two characteristic waves, both circularly polarized and with different propagation constants: β_+ corresponding to the right circularly polarized wave and β_- to the left circularly polarized wave; only when $\xi = \zeta$ linearly polarized fields are admissible since $\beta_+ = \beta_-$ and the biisotropic medium is "nonactive" [7].

D. Application to Chiral Media

For reciprocal biisotropic media, which will be referred to simply as chiral media, one may write [1], [7]

$$\xi = -\zeta = -j\chi \tag{27}$$

where χ is real (positive or negative) for lossless media.

As already pointed out, in this reciprocal case the linear-operator formalism which has been developed is reduced to a *self-adjoint* problem. Moreover, from (26) and (27), one obtains ($s = 1, 2$)

$$\tau_s = \pm jy \qquad y = \sqrt{\frac{\epsilon}{\mu}} \tag{28a}$$

$$\sigma_s^2 = \beta_\pm^2 - \beta^2 \quad \beta_\pm = \sqrt{\epsilon\mu} \pm \chi \tag{28b}$$

where, once more, the *plus* (resp. *minus*) sign corresponds to $s = 1$ (resp. $s = 2$).

III. Radiation Modes of a Grounded Chiroslabguide

As an example of application of the linear-operator formalism, the grounded chiroslabguide depicted in Fig. 1 will be analyzed. Since the grounded chiroslabguide is an *open structure* extending from the perfectly conducting plate at $x' = 0$ to $x' = \infty$, one should remark that, for completeness, the radiation (or pseudosurface) modes should be included [20]. Moreover, the radiation modes do not actually belong to the domain of the operator: indeed they are improper eigenfunctions [9].

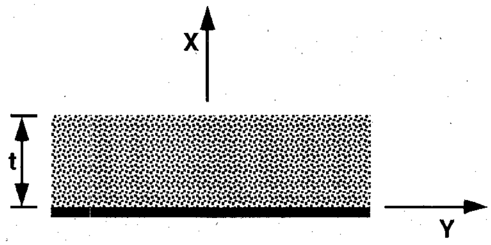

Fig. 1. Grounded chiroslabguide (conductor-backed chiral slabguide). The slab, characterized by the three dimensionless parameters (ϵ, μ, χ), has thickness t and is in contact with air.

As the surface modes of this open waveguide were already analyzed by the authors elsewhere [11] as well as by Engheta and Pelet in [15], only the radiation modes will be thereafter considered.

One should note, however, that the surface (resp. radiation) modes of a grounded chiroslabguide are not, at the same time, surface (resp. radiation) modes of a corresponding symmetric chiroslabguide with double thickness as in the isotropic case. In fact, due to the general characteristics of spatial inversion in bianisotropic media, the *conventional image theory* is no longer applicable to biisotropic media [1]. Indeed, the modes of the grounded chiroslabguide depicted in Fig. 1 are the same (for $x' > 0$) as for an asymmetric chiroslabguide where the region $-t' < x' < 0$ is filled with the mirror-conjugate chiral medium characterized by (ϵ, μ, $-\chi$)[1].

A. Eigenfunctions for the Grounded Chiroslabguide

According to (23), (24) and (28), one has for $0 < x' < t'$,

$$E_y = \psi_1 + \psi_2 \tag{29a}$$

$$\mathcal{H}_y = jy(\psi_1 - \psi_2) \tag{29b}$$

where

$$\psi_1 = A[\sin(\sigma_1 x') - Q\cos(\sigma_1 x')] \tag{30a}$$

$$\psi_2 = A[R\sin(\sigma_2 x') + Q\cos(\sigma_2 x')]. \tag{30b}$$

The form of (30) automatically guarantees $E_y = 0$ for $x' = 0$.

Imposing the other boundary condition at the perfectly conducting plate at $x' = 0$, i.e., $E_z = 0$, one obtains

$$R = \frac{\sigma_1}{\sigma_2}\frac{\beta_-}{\beta_+}. \tag{31}$$

For the air region, i.e., for $x' > t'$, one gets

$$E_y = \alpha_1 A\{\cos[\rho(x' - t')] + B\sin[\rho(x' - t')]\} \tag{32a}$$

$$\mathcal{H}_y = j\alpha_2 A\{\cos[\rho(x' - t')] + B\sin[\rho(x' - t')]\} \tag{32b}$$

[1]Obviously that, for $x' < -t'$, the medium of this asymmetric guide is the air—as for $x' > t'$.

where

$$\rho^2 = 1 - \beta^2. \tag{33}$$

For $0 < \rho < 1$ one has (fast) propagating radiation modes with $0 < \beta < 1$; for $\rho > 1$ one has evanescent radiation modes with an imaginary β; for $\rho = 1$, $\beta = 0$ and hence there is only an oscillation without propagation. According to (32), E_y and $\mathcal{H}_y$ have the same standing-wave behavior in the air, i.e., one has ($x' > t'$)

$$\mathcal{H}_y = j \frac{\alpha_2}{\alpha_1} E_y. \tag{34}$$

In Appendix III expressions for the coefficients Q, α_1, α_2 and B are given. Coefficient A will be evaluated next according to the orthogonality relation.

B. Orthogonality Relation for the Grounded Chiroslabguide

Using a similar procedure to the one presented in [10], the orthogonality relation for the radiation modes can be easily derived. Therefore, omitting the details for the sake of brevity, one can state the following orthogonality relation:

$$\mathcal{A}(\rho, \rho') + j\mathcal{B}(\rho, \rho') = \delta(\rho - \rho') \tag{35}$$

where

$$\mathcal{A} = \int_0^\infty \frac{1}{\Delta(x')} [\epsilon(x') E_y(x', \rho) E_y(x', \rho') - \mu(x') \mathcal{H}_y(x', \rho) \mathcal{H}_y(x', \rho')] \, dx' \tag{36a}$$

$$\mathcal{B} = \int_0^\infty \frac{\chi(x')}{\Delta(x')} [E_y(x', \rho) \mathcal{H}_y(x', \rho') + \mathcal{H}_y(x', \rho) E_y(x', \rho')] \, dx' \tag{36b}$$

with

$$\epsilon(x') = \epsilon U(x') + (1 - \epsilon) U(x' - t') \tag{37a}$$

$$\mu(x') = \mu U(x') + (1 - \mu) U(x' - t') \tag{37b}$$

$$\chi(x') = \chi[U(x') - U(x' - t')] \tag{37c}$$

$$\Delta(x') = \epsilon(x') \mu(x') - \chi^2(x') \tag{37d}$$

and where $U(x')$ is the step function

$$U(x') = \begin{cases} 1, & \text{for } x' > 0 \\ 0, & \text{for } x' < 0. \end{cases} \tag{38}$$

Obviously that, since $\chi(x') = 0$ for $x' > t'$, the upper limit of the integral in (36b) can be replaced by t'. One should also note that—as far as the authors are aware—this orthogonality relation has never been presented in the literature.

Taking into consideration the expressions for $E_x(x', \rho)$ and $\mathcal{H}_x(x', \rho)$ presented in Appendix I, the orthogonality relation (35) can be rewritten as

$$\int_0^\infty [E_x(x', \rho) \mathcal{H}_y(x', \rho') + \mathcal{H}_x(x', \rho) E_y(x', \rho')] \, dx' = -\beta\delta(\rho - \rho') \tag{39}$$

which is formally similar to (18).

Accordingly, for A in (30) and (32), one should write

$$A = \frac{1}{\alpha_1} \sqrt{\frac{2\mu}{\pi(\epsilon + \mu)(1 + B^2)}} \tag{40}$$

since $\delta(\rho + \rho') = 0$.

One should finally note that, for $\rho = 0$, there is no radiation field. In fact, according to (40) as well as to the expression for B (Appendix III) one has $A \to 0$ when $\rho \to 0$.

IV. Conclusion

A linear-operator formalism for the analysis of inhomogeneous biisotropic waveguides was developed. The original and adjoint waveguides were described by eigenvalue equations related, respectively, to a 2×2 matrix differential operator and its transpose. Accordingly, a biorthogonality relation for the hybrid modes, which involves the two-vector eigenfunctions of both the original and adjoint waveguides was derived. In the reciprocal biisotropic (or chiral) case, the original and adjoint waveguides are identical, therefore leading to a self-adjoint problem. For the particular case of biisotropic multilayered waveguides with homogeneous layers, the general framework based on the theory of linear operators is reduced to a simple 2×2 matrix eigenvalue problem.

As an example of application, a general analysis of the continuous spectrum of a grounded chiroslabguide was also developed. Namely, a new orthogonality relation for the radiation modes of this open chirowaveguide was presented.

Finally one should remark that the usefulness of this formalism is beyond its own theoretical interest as, e.g., in a building-block approach. In fact, it is likely to become a valuable tool in a mode matching procedure for the study of more complex structures (e.g., step discontinuities on planar chirowaveguides) where it is necessary to describe the field components in each subregion in terms of a complete set of transverse mode functions.

Appendix I
Field Components of the Hybrid Modes

According to (1)–(5) and taking E_y and $\mathcal{H}_y$ as the supporting field components, one obtains

$$E_x = \frac{\beta}{\Delta} (\xi E_y + \mu \mathcal{H}_y) \tag{A1a}$$

$$\mathcal{H}_x = -\frac{\beta}{\Delta} (\epsilon E_y + \zeta \mathcal{H}_y) \tag{A1b}$$

and

$$E_z = -j \frac{1}{\Delta} (\xi \partial_{x'} E_y + \mu \partial_{x'} \mathcal{H}_y) \quad \text{(A2a)}$$

$$\mathcal{H}_z = j \frac{1}{\Delta} (\epsilon \partial_{x'} E_y + \zeta \partial_{x'} \mathcal{H}_y) \quad \text{(A2b)}$$

where Δ was introduced in (10).

Appendix II
Proof That $\overline{\mathcal{L}}^a$ Is the Adjoint Operator of $\overline{\mathcal{L}}$

If $\boldsymbol{u} = [u_1, u_2]^T \in D$ and $\boldsymbol{u}^a = [u_1^a, u_2^a]^T \in D^a$ one has to prove that

$$J = \langle \overline{\mathcal{L}} \cdot \boldsymbol{u}, \boldsymbol{u}^a \rangle - \langle \boldsymbol{u}, \overline{\mathcal{L}}^a \cdot \boldsymbol{u}^a \rangle = 0. \quad \text{(A3)}$$

Due to the definition (14) and according to (8) and (13), one obtains—after cancelling the identical terms—the following expression:

$$J = \sum_{i=1}^{4} J_i \quad \text{(A4)}$$

where

$$J_1 = \int_I \left[u_1^a \, \partial_{x'} \left(\frac{\epsilon}{\Delta} \partial_{x'} u_1 \right) - u_1 \, \partial_{x'} \left(\frac{\epsilon}{\Delta} \partial_{x'} u_1^a \right) \right] dx' \quad \text{(A5)}$$

$$J_2 = \int_I \left[u_2 \, \partial_{x'} \left(\frac{\mu}{\Delta} \partial_{x'} u_2^a \right) - u_2^a \, \partial_{x'} \left(\frac{\mu}{\Delta} \partial_{x'} u_2 \right) \right] dx' \quad \text{(A6)}$$

$$J_3 = \int_I \left[u_1^a \, \partial_{x'} \left(\frac{\zeta}{\Delta} \partial_{x'} u_2 \right) - u_2 \, \partial_{x'} \left(\frac{\zeta}{\Delta} \partial_{x'} u_1^a \right) \right] dx' \quad \text{(A7)}$$

$$J_4 = \int_I \left[u_1 \, \partial_{x'} \left(\frac{\xi}{\Delta} \partial_{x'} u_2^a \right) - u_2^a \, \partial_{x'} \left(\frac{\xi}{\Delta} \partial_{x'} u_1 \right) \right] dx'. \quad \text{(A8)}$$

Using integration by parts, one gets

$$J_1 = \left[\frac{\epsilon}{\Delta} u_1^a \, \partial_{x'} u_1 \right]_I - \left[\frac{\epsilon}{\Delta} u_1 \, \partial_{x'} u_1^a \right]_I \quad \text{(A9)}$$

$$J_2 = \left[\frac{\mu}{\Delta} u_2 \, \partial_{x'} u_2^a \right]_I - \left[\frac{\mu}{\Delta} u_2^a \, \partial_{x'} u_2 \right]_I \quad \text{(A10)}$$

$$J_3 = \left[\frac{\zeta}{\Delta} u_1^a \, \partial_{x'} u_2 \right]_I - \left[\frac{\zeta}{\Delta} u_2 \, \partial_{x'} u_1^a \right]_I \quad \text{(A11)}$$

$$J_4 = \left[\frac{\xi}{\Delta} u_1 \, \partial_{x'} u_2^a \right]_I - \left[\frac{\xi}{\Delta} u_2^a \, \partial_{x'} u_1 \right]_I \quad \text{(A12)}$$

with $[f]_I = f(x_2') - f(x_1')$ and where x_1' (resp. x_2') is the lower (resp. upper) limit of interval I.

One can easily see that, for any class of I, $J_i = 0$ for $1 \le i \le 4$, and hence, according to (A4), $J = 0$ (q.e.d.). In fact, if an electric wall is placed at x_k' ($k = 1, 2$), one should have

$$u_1(x_k') = \xi(x_k') \, \partial_{x'} u_1(x_k') + \mu(x_k') \, \partial_{x'} u_2(x_k') = 0 \quad \text{(A13a)}$$

$$u_1^a(x_k') = -\zeta(x_k') \, \partial_{x'} u_1^a(x_k') + \mu(x_k') \, \partial_{x'} u_2^a(x_k') = 0. \quad \text{(A13b)}$$

On the other hand, if a magnetic wall is placed at x_k', one should have instead

$$u_2(x_k') = \epsilon(x_k') \, \partial_{x'} u_1(x_k') + \zeta(x_k') \, \partial_{x'} u_2(x_k') = 0 \quad \text{(A14a)}$$

$$u_2^a(x_k') = \epsilon(x_k') \, \partial_{x'} u_1^a(x_k') - \xi(x_k') \, \partial_{x'} u_2^a(x_k') = 0. \quad \text{(A14b)}$$

Finally, if the hybrid mode is a surface wave, one should have ($s = 1, 2$)

$$u_s(\pm\infty) = u_s^a(\pm\infty) = 0 \quad \text{(A15a)}$$

$$\partial_{x'} u_s(\pm\infty) = \partial_{x'} u_s^a(\pm\infty) = 0. \quad \text{(A15b)}$$

Appendix III
Coefficients of the Radiation Modes

Imposing the continuity of the tangential field components at the interface ($x' = t'$), coefficients Q, α_1, α_2 and B can be determined.

Therefore

$$Q = \frac{(\epsilon - \mu) \sin(\sigma_1 t') - (\epsilon + \mu) R \sin(\sigma_2 t')}{(\epsilon - \mu) \cos(\sigma_1 t') + (\epsilon + \mu) \cos(\sigma_2 t')} \quad \text{(A16)}$$

and

$$\alpha_1 = y \alpha_2 \quad \text{(A17)}$$

where

$$\alpha_1 = [\sin(\sigma_1 t') + R \sin(\sigma_2 t')] - Q[\cos(\sigma_1 t') - \cos(\sigma_2 t')]. \quad \text{(A18)}$$

Finally,

$$B = \frac{y \sigma_2 \beta_+}{\alpha_1 \rho \Delta} \{Q[R \sin(\sigma_1 t') + \sin(\sigma_2 t')] + R[\cos(\sigma_1 t') - \cos(\sigma_2 t')]\}. \quad \text{(A19)}$$

Introducing, according to (A19), coefficient B_0 such that

$$B = \frac{B_0}{\rho} \quad \text{(A20)}$$

one can easily see that

$$\lim_{\rho \to 0} \{B \sin [\rho(x' - t')]\} = B_0(x' - t') \quad \text{(A21)}$$

in (32a) and (32b).

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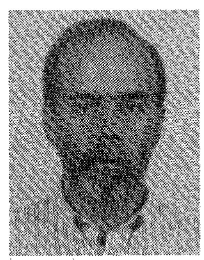

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